GEOMETRY OF BANACH SPACES WITH PROPERTY β

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To the memory of A. Plans

ABSTRACT

We prove that every Banach space can be isometrically and 1-complementably embedded into a Banach space which satisfies property β and has the same character of density. Then we show that, nevertheless, property β satisfies a hereditary property. We study strong subdifferentiability of norms with property β to characterize separable polyhedral Banach spaces as those admitting a strongly subdifferentiable β norm. In general, a Banach space with such a norm is polyhedral. Finally, we provide examples of non-reflexive spaces whose usual norm fails property β and yet it can be approximated by norms with this property, namely $(L_1[0,1], \|\cdot\|_1)$ and $(C(K), \|\cdot\|_\infty)$ where K is a separable Hausdorff compact space.

1. Introduction

A Banach space X with norm $\|\cdot\|$ and dual space X^* has property β if there exists a system $\{x_i, x_i^*\}_{i\in I} \subset X \times X^*$ and a real number $0 \le \varepsilon < 1$ satisfying

$$x_i^*(x_i) = 1 = \|x_i\| = \|x_i^*\|, \quad |x_i^*(x_j)| \le \varepsilon, \quad i \neq j, \tag{1}$$

$$\|x\| = \sup_{i\in I} |x_i^*(x)|. \tag{2}$$

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In this case, the norm $\|\cdot\|$ is said to have property (β, ε) or simply β if we do not need to specify the parameter ε.

Property β was introduced by J. Lindenstrauss [13] in the study of norm attaining operators. He used this property as a sufficient condition for a Banach space X to have the so-called property B: the set of norm attaining operators from any Banach space into X is dense in the set of all bounded operators. The first result concerning renormings of Banach spaces with property β was proved by J. Partington [15] in a remarkable paper. He showed that every Banach space can be equivalently renormed to have property β. After these pioneering works, many authors have turned their attention to the study of property β. C. Finet and W. Schachermayer [4] introduced property β in the strong sense. On the other hand, M. D. Acosta, F. J. Aguirre and R. Payá [1] considered property quasi-β, a weakness of β still implying B and stable under c_0-sums.

We are concerned in this note with the geometry of spaces with property β. In Section 2, we prove that every Banach space can be isometrically and 1-complementably embedded into a Banach space with property β, and having the same character of density. Then we show that, nevertheless, property β satisfies a special kind of hereditary property.

In Section 3, we deal with β norms which are strongly subdifferentiable (β-SSD norms, for short). We first extend the result of C. Franchetti [7] related to the strong subdifferentiability of the cannonical sup-norm $\|\cdot\|_\infty$ on ℓ_∞ to every norm β. Using this fact we prove that a β-SSD norm is polyhedral. Conversely, we prove that every separable polyhedral space admits a β-SSD norm, thus obtaining a new characterization of these spaces. We provide examples of non-separable polyhedral spaces admitting an equivalent β-SSD norm. This is the case of the preduals of Lorentz sequence spaces $d_*(w, 1, \Gamma)$, for every infinite set Γ. Besides these results, however, the question of whether every polyhedral Banach space admits a β-SSD norm still remains open.

The last section is devoted to approximation by β norms. W. Schachermayer showed that norms with property α or β are dense in every superreflexive space [16]. He gave an example, namely ℓ_1, of a non-reflexive space whose usual norm does not have property β and, however, can be approximated by norms with this property. We provide analogous examples of this situation, namely $(C(K), \|\cdot\|_\infty)$ (where K is a separable compact Hausdorff space) and $(L_1[0,1], \|\cdot\|_1)$.

2. Renorming results

Throughout this paper, we consider only infinite dimensional Banach spaces over the reals. Given a Banach space X, we denote by $B(X)$ the closed unit ball, by $S(X)$ the unit sphere and by $\operatorname{dens} X$ its character of density. Our first result concerns embeddability of Banach spaces into spaces with property β.

PROPOSITION 2.1: *Every Banach space $(X, \|\cdot\|)$ can be isometrically and 1-complemented embedded into a Banach space Y with property $(\beta, 1/2)$ and $\operatorname{dens} Y = \operatorname{dens} X$. Therefore, X is isometric to a quotient of a Banach space with property β.*

Proof: Let $\operatorname{dens} X = \gamma$, $I = [0, \gamma)$ and consider $Y = X \oplus c_0(I)$ equipped with the norm $\|(x,t)\| = \max\{\|x\|, \|t\|_\infty\}$ for $x \in X$ and $t \in c_0(I)$. Note that $\operatorname{dens} X = \operatorname{dens} Y$. Take a set $\{x_\alpha\}_{\alpha<\gamma}$ dense in $S(X)$ and corresponding $\{x_\alpha^*\}_{\alpha<\gamma} \subset S(X^*)$ so that $x_\alpha^*(x_\alpha) = 1$. Clearly, $\|x\| = \sup_{\alpha<\gamma} x_\alpha^*(x)$ for every $x \in X$. Let $\{e_\alpha\}_{\alpha<\gamma}$ be the unit vectors of the canonical basis of $c_0(I)$ and $\{e_\alpha^*\}_{\alpha<\gamma}$ the associated functionals. Set $y_{\alpha 1} = e_\alpha + x_\alpha$, $y_{\alpha 2} = e_\alpha - x_\alpha$, $y_{\alpha 1}^* = e_\alpha^* + x_\alpha^*$ and $y_{\alpha 2}^* = e_\alpha^* - x_\alpha^*$. Then $y_{\alpha 1}^*(y_{\alpha 1}) = y_{\alpha 2}^*(y_{\alpha 2}) = 2$, $y_{\alpha i}^*(y_{\alpha j}) = 0$, for $i \neq j$ and $|y_{\alpha i}^*(y_{\beta j})| \leq 1$, for $\alpha \neq \beta$. Then, a straight verification shows that the equivalent norm on X defined by

$$|(x,t)| = \sup_{\alpha<\gamma,\ i=1,2} \{|y_{\alpha i}^*(x,t)|\}$$

has property $(\beta, 1/2)$ with respect to the system $\{y_{\alpha i}^*, y_{\alpha i}\}$, $\alpha < \gamma$, $i = 1, 2$ and the rest of the required properties. ∎

One noteworthy conclusion to be drawn from Proposition 2.1 is the fact that property β does not entail any condition on the geometry of the subspaces, even though they were complemented. However, regarding the space Y and its subspace $c_0(I)$ both having property β, one may wonder if this property is actually hereditary in some sense. The next result shows that it is the case.

PROPOSITION 2.2: *Let $(X, \|\cdot\|)$ be a Banach space satisfying property β with respect to the system $\{x_i, x_i^*\}_{i \in I} \subset X \times X^*$ and let $Y \subset X$ be a (closed) subspace of X. There is $J \subset I$ with $\operatorname{card} J = \operatorname{dens} Y$ such that, if we consider $Z = \overline{\operatorname{span}}\,(Y \cup \{x_j\}_{j \in J})$, then $(Z, \|\cdot\|)$ satisfies property β with respect to the system $\{x_j, x_j^*\}_{j \in J} \in Z \times Z^*$.*

Proof: Take a set $\{d_\alpha^1\}_{\alpha \in \Gamma_1}$ dense in $S(Y)$ (the unit sphere of Y with the restricted norm) with $\operatorname{card} \Gamma_1 = \operatorname{dens} Y$. For each $\alpha \in \Gamma_1$ there is a sequence $\{x_{\alpha k}^{*1}\}_{k=1}^\infty \subset \{x_i^*\}_{i \in I} \subset X^*$ such that $\lim_k |x_{\alpha k}^{*1}(d_\alpha^1)| = 1$. For each $\alpha \in \Gamma_1$ and $k \in \mathbb{N}$, choose $\{x_{\alpha k}^1\} \subset \{\pm x_i\}_{i \in I}$ so that $x_{\alpha k}^{*1}(x_{\alpha k}^1) = 1$ and define

$$Y_2 = \overline{\operatorname{span}} \left\{ \bigcup_{\alpha \in \Gamma_1} \{x_{\alpha k}^1\}_{k=1}^\infty \cup Y \right\}.$$

Assume that Y_n has been defined. Again, if $\{d_\alpha^n\}_{\alpha\in\Gamma_n}$ is a dense subset of $S_{\|\cdot\|}\cap Y_n$ with $\operatorname{card}\Gamma_n = \operatorname{dens} Y_n$, for each $\alpha \in \Gamma_n$ there is a sequence $\{x_{\alpha k}^{*n}\}_{k=1}^\infty \subset \{x_i^*\}_{i\in I} \subset X^*$ satisfying $\lim_k |x_{\alpha k}^{*n}(d_\alpha^n)| = 1$. For each $\alpha \in \Gamma_n$ and $k \in \mathbb{N}$, choose $\{x_{\alpha k}^n\} \subset \{\pm x_i\}_{i\in I}$ such that $x_{\alpha k}^{*n}(x_{\alpha k}^n) = 1$ and define

$$Y_{n+1} = \overline{\operatorname{span}}\left\{\bigcup_{\alpha\in\Gamma_n}\{x_{\alpha k}^n\}_{k=1}^\infty \cup Y_n\right\}.$$

So far we have a sequence of subspaces $\{Y_n\}_{n=1}^\infty \subset X$ satisfying $Y_n \subset Y_{n+1}$ and $\operatorname{dens} Y_n = \operatorname{dens} Y_{n+1}$. Set

$$Z = \overline{\operatorname{span}}\left(\bigcup_{n=1}^\infty Y_n\right).$$

Clearly, $Y \subset Z$ and $\operatorname{dens} Z = \operatorname{dens} Y$. In order to finish the proof we must verify that

$$\{x_{\alpha k}^n,\ x_{\alpha k}^{*n}|_Z,\ \alpha \in \Gamma_n,\ k \in \mathbb{N}\} \subset Z \times Z^*$$

satisfies property (2) since property (1) is trivially fulfilled. For this purpose, take $z \in Z$, $\delta > 0$, and $y \in \operatorname{span}(\bigcup_{n=1}^\infty Y_n)$ such that $\|z-y\| < \frac{1}{n}$. Then, there is $n_0 \in \mathbb{N}$ so that $y \in Y_{n_0}$. We can find $d_{\alpha_0}^{n_0} \in \{d_\alpha^{n_0}\}_{\alpha\in\Gamma_{n_0}}$ satisfying $\|y - d_{\alpha_0}^{n_0}\| < \frac{1}{n}$. By definition of $\{x_{\alpha_0 k}^{*n_0}\}_{k=1}^\infty$, we can pick $k_0 \in \mathbb{N}$ so that $x_{\alpha_0 k_0}^{*n_0}$ verifies $|x_{\alpha_0 k_0}^{*n_0}(d_{\alpha_0}^{n_0})| > 1 - \frac{1}{n}$. Hence

$$|x_{\alpha_0 k_0}^{*n_0}(z)| \geq |x_{\alpha_0 k_0}^{*n_0}(d_{\alpha_0}^{n_0})| - |x_{\alpha_0 k_0}^{*n_0}(y - d_{\alpha_0}^{n_0})| - |x_{\alpha_0 k_0}^{*n_0}(z-y)| \geq 1 - 3(1/n),$$

thus entailing that

$$\|z\| = \sup\left\{|x_{\alpha k}^{*n}(z)|,\ n \in \mathbb{N}, \alpha \in \Gamma_n,\ k \in \mathbb{N}\right\}$$

for each $z \in Z$ and the proof is finished. ∎

3. Strong subdifferentiable β norms: a new characterization of separable polyhedral Banach spaces

The norm $\|\cdot\|$ on X is said to be strongly subdifferentiable (SSD, for short) at x if the one-sided limit

$$\lim_{t\to 0^+} \frac{1}{t}\left(\|x+th\| - \|x\|\right)$$

exists uniformly on $h \in S(X)$. This non-smooth extension of Fréchet differentiability has been encountered by many authors, frequently under different definitions, and has found interesting applications (see [8] and [10] for the references and for recent results on this topic). It is well known [7] that the canonical sup-norm $\|\cdot\|_\infty$ on $\ell_\infty(\Gamma)$ is SSD at $x=(x_i)$ if and only if x belongs to the set $\mathcal{S}(\Gamma)=\{x\in\ell_\infty(\Gamma)\colon \|x\|_\infty>\sup\{|x_i|\colon |x_i|\neq\|x\|_\infty\}$. Consequently, $\|\cdot\|_\infty$ is SSD at each point of $c_0(\Gamma)$. We will say that a norm is β-SSD provided it has property β and it is SSD at each non-zero point of the space.

LEMMA 3.1: *The norm $\|\cdot\|$ on X is β-SSD if and only if there exists a system $\{x_i,x_i^*\}_{i\in\Gamma}\subset X\times X^*$ and $0\le\varepsilon<1$ satisfying (1), (2) and $T(x)=(x_i^*(x))\in\mathcal{S}(\Gamma)$ for each $x\in X$.*

Proof: Since T is an isometry between $(X,\|\cdot\|)$ and $(T(X),\|\cdot\|_\infty)$, the sufficiency follows from [7]. To prove the necessity, take $x\in S(X)$ and assume that there exists a sequence $\{x_k^*\}\subset\{\pm x_i^*\}_{i\in\Gamma}$ such that $0<x_k^*(x)<1$ and $\lim_k x_k^*(x)=1$. Then, $\lim_{t\to0^+}\frac{1}{t}\left(\|x+tx_k\|-1\right)$ is not uniform on k. Indeed,

$$\|x+tx_k\|=\sup|x_i^*(x)+tx_i^*(x_k)|\le\max\{1+t\varepsilon,x_k^*(x)+t\},$$

so for t small enough we know that $\|x+tx_k\|\le1+t\varepsilon$ and hence that

$$\lim_{t\to0^+}\frac{1}{t}\left(\|x+tx_k\|-1\right)\le\varepsilon.$$

On the other hand, if $t\ge(1-\varepsilon)^{-1}(1-x_k^*(x))$ then $x_k^*(x)+t\ge1+t\varepsilon$ and $\|x+tx_k\|=x_k^*(x)+t$. Considering now $t=2(1-\varepsilon)^{-1}(1-x_k^*(x))$ we have

$$\frac{\|x+tx_k\|-1}{t}=\frac{\varepsilon+1}{2}$$

although, obviously, $\lim_k(1-\varepsilon)^{-1}(1-x_k^*(x))=0$. ∎

Recall that a subset $B\subset B(X^*)$ is said to be a **boundary** [9, 11] if for every $x\in X$ there exists $f\in B$ such that $f(x)=\|x\|$. The space X is **polyhedral** [12] if the unit ball of any of its finite dimensional subspaces is a polyhedron. These spaces have been intensively studied by V. Fonf (see [5], [6]) who, among many other things, proved the following relationship with boundaries: (A) if a Banach space has a countable boundary $\{f_n\}$, then this space is polyhedral under the equivalent norm $\|x\|=\sup\{(1+\varepsilon_n)f_n(x),n\in\mathbb{N}\}$ where $\{\varepsilon_n\}$ is a decreasing sequence of positive real numbers with $\lim_n\varepsilon_n=0$; (B) conversely, given a polyhedral Banach space of density α, there exists a boundary B of cardinality α such that for every $f\in B$ the face $\{x\in S(X):f(x)=1\}$ has non-empty interior in the hyperplane $\{x\in X\colon f(x)=1\}$. He also proved that each polyhedral space contains a copy of c_0 [5].

PROPOSITION 3.2: *Every β-SSD norm is polyhedral.*

Proof: Let us consider a β-SSD norm $\|\cdot\|$ on a Banach space X with associated system $\{x_i, x_i^*\}_{i\in I}$. Notice that from Lemma 3.1, the set $B = \{\pm x_i^*\}_{i\in I}$ is a boundary and also

$$\delta_x = 1 - \sup\{|x_i^*(x)|: |x_i^*(x)| \neq 1\} > 0$$

for every $x \in S(X)$. Consider now $y \in S(X)$ with $\|x-y\| < \delta_x$, and $x^* \in B$ with $x^*(y) = 1$. Clearly $x^*(x) = 1$, thus implying that the segment $[x, y]$ is included in $S(X)$. This yields in particular that the set of extreme points of the unit sphere of any (closed) subspace of X has no accumulation points. Therefore, the restricted norm on a finite dimensional subspace has a finite number of extreme points in its unit sphere. ∎

A particular case of the above Proposition was proved in [3]. We come now to the main result of this section which, for separable spaces and up to an equivalent renorming, is the converse of Proposition 3.2. It yields a new and surprising characterization of separable polyhedral Banach spaces.

We say that a norm $\|\cdot\|$ with a boundary $\{f_n\}$ satisfies $(*)$ whenever for each $x \in X\setminus\{0\}$ there is $n_0(x) \in \mathbb{N}$ and $\alpha(x) > 0$ such that if $n \geq n_0(x)$, then $|f_n(x)| \leq \|x\| - \alpha(x)$. Therefore, this norm is SSD. Moreover, it can be proved, by an easy compacity argument, that for every finite dimensional subspace F, there is $N \in \mathbb{N}$ so that $\|x\| = \sup\{f_j(x): j \leq N\}$, if $x \in F$. Thus, the norm $\|\cdot\|$ is also polyhedral. Observe, for instance, that the norm $\|\cdot\|$ exhibited in (A) satisfies $(*)$. Recall that a norm $|\cdot|$ is K-equivalent to $\|\cdot\|$ if $\|x\| \leq |x| \leq K\|x\|$, for every element x in the space.

PROPOSITION 3.3: *Let X be a separable polyhedral Banach space. Then, for any equivalent norm $\|\cdot\|$ and $K > 3$ there is a K-equivalent and (β,ε)-SSD norm $|\cdot|$ (even with property $(*)$ and ε depending only on K).*

Proof: It is proved in [2] that every equivalent norm in a separable polyhedral Banach space can be uniformly approximated on bounded sets by norms $N(\cdot)$ satisfying $(*)$, so we can assume that $\|\cdot\|$ also satisfies $(*)$.

Furthermore, by [5] and [14, p. 97], we obtain that for every $\varepsilon > 0$ the space $(X, \|\cdot\|)$ contains a subspace ε-isometric to c_0 with its usual norm $\|\cdot\|_\infty$ and then, using [14, p. 106] for every $\varepsilon > 0$, there is a projection $p : (X, \|\cdot\|) \longrightarrow (c_0, \|\cdot\|_\infty)$ of norm less than or equal to $2+\varepsilon$. For these reasons, we may assume that:

1. if we denote by (e_n) the canonical basis of c_0 and consider them as elements of X, the norm $\|e_n\| \leq 1+\varepsilon$;

2. if we denote by (e_n^*) the canonical basis of ℓ_1, the elements of X^* defined by $z_n^* = e_n^* \circ p$ have norm $\|z_n^*\|^* \leq 2 + \varepsilon$.

We shall prove that for $\varepsilon > 0$ small enough and $M = 1 + 4\sqrt{\varepsilon}$, the new equivalent norm $|\cdot|$, defined as

$$|x| = \sup_{n \in \mathbb{N},\, \theta = \pm 1} |g_{n,\theta}(x)|$$

where $g_{n,\theta} = f_n + \theta M z_n^*$, is β–SSD. Clearly $\|x\| \leq |x| \leq (1 + 2M + \varepsilon M)\|x\|$. The fact that $\lim_n z_n^* = 0$ in the weak* topology and property $(*)$ implies that for every $\theta = \pm 1$ and $x \in X \setminus \{0\}$,

$$\limsup_n |g_{n,\theta}(x)| \leq \limsup_n |f_n(x)| \leq \|x\| - \alpha(x) \leq |x| - \alpha(x).$$

Therefore, $(g_{n,\theta})$ is a countable boundary satisfying also $(*)$ and thus $|\cdot|$ is polyhedral and SSD. In order to prove that $|\cdot|$ is β, take $(x_n) \subset S_{\|\cdot\|}$, the unit sphere of $\|\cdot\|$, so that $f_n(x_n) = 1$ (if for some k such a point does not exist we can remove f_k as an element of the boundary $\{f_n\}$). Passing to subsequences of (e_n) and (z_n^*) if necessary, we may assume that $|f_n(e_n)| \leq \varepsilon$ and $|z_n^*(x_n)| \leq \varepsilon$. If we associate to each $g_{n,\theta}$ the element $y_{n,\theta} = y_n + \theta M' e_n \in X$ with $M' = \frac{1}{\sqrt{\varepsilon}}$ we obtain

$$g_{n,\theta}(y_{n,\theta}) \geq 5 + \frac{1}{\sqrt{\varepsilon}} - \sqrt{\varepsilon}(1 + \sqrt{\varepsilon} + 4\varepsilon),$$
$$|g_{n,\theta}(y_{n,-\theta})| \leq 3 + \frac{1}{\sqrt{\varepsilon}} + \sqrt{\varepsilon}(1 + \sqrt{\varepsilon} + 4\varepsilon),$$
$$|g_{n,\theta}(y_{m,\theta'})| \leq 3 + \frac{1}{\sqrt{\varepsilon}} + \sqrt{\varepsilon}(9 + \sqrt{\varepsilon} + 4\varepsilon), \qquad \text{if } n \neq m,$$

and thus for $\varepsilon > 0$ small enough we deduce that $|\cdot|$ is β. ∎

Notice that the constant $K > 3$ is the best known to approximate an arbitrary norm by β norms in non-superreflexive Banach spaces [15].

COROLLARY 3.4: *A separable Banach space is polyhedral if and only if it can be (equivalently) renormed with a β-SSD norm.*

The situation in non-separable Banach spaces seems to be not so clear. A "non-separable" version of the proof of Proposition 3.3 applies to find β-SSD norms in some non-separable polyhedral Banach spaces. We will finish this section by exhibiting examples of classical non-separable polyhedral Banach spaces admitting equivalent β-SSD norms.

PROPOSITION 3.5: *Let $X = \left(\bigoplus_{i\in\Gamma} X_i\right)_{c_0}$, where Γ is an infinity cardinal and X_i are separable polyhedral Banach spaces. Then X has a β-SSD norm.*

Proof: Take $K > 3$ and let, for each $i \in \Gamma$, $\{x_{ni}, f_{ni}\}_{n=1}^{\infty}$ be a system (which has come from Proposition 3.3) such that the norm on X_i,

$$|u| = \sup\{|x_{ni}^*(u)| : 1 \le n < \infty\}, \quad u \in X_i,$$

is K-equivalent to the original one and β-SSD. It is clear that the equivalent norm on X,

$$|x| = \sup\{|x_{ni}^*(x)| : 1 \le n < \infty, i \in \Gamma\},$$

is K-equivalent and β-SSD too. ∎

Let us consider now the infinite predual Lorentz sequence spaces $d_*(w,1,\Gamma)$, for any infinite set Γ. Recall that (w_n) is a non-increasing sequence of positive numbers such that $w_1 = 1$, $\lim_{n\to\infty} w_n = 0$ and $\sum_n w_n = \infty$. The Banach space $d_*(w,1,\Gamma)$ consists of all points $x = (x_i)_{i\in\Gamma} \in c_0(\Gamma)$ so that if $|x_{i_1}| \ge |x_{i_2}| \ge |x_{i_3}| \ge \cdots$ is the non-increasing rearrangement of the non-zero coefficients of $(|x_i|)$, then

$$\lim_n \frac{\sum_1^n |x_{i_k}|}{\sum_1^n w_k} = 0.$$

The norm of a point $x \in d_*(w,1,\Gamma)$ is

$$\|x\| = \sup_n \frac{\sum_1^n |x_{i_k}|}{\sum_1^n w_k}.$$

PROPOSITION 3.6: *For any Γ, the Banach space $d_*(w,1,\Gamma)$ admits a β-SSD norm.*

Proof: Let us denote by (e_i) and (e_i^*) the canonical bases of $d_*(w,1,\Gamma)$ and its dual $d(w,1,\Gamma)$, respectively. Recall that the family

$$\mathcal{F} = \left\{ \frac{\sum_1^n \varepsilon_k e_{i_k}^*}{\sum_1^n w_k} : n \in \mathbb{N},\ i_k \in \Gamma,\ i_k \ne i_m,\ \text{if } k \ne m \text{ and } \varepsilon_k = \pm 1 \right\}$$

is a boundary of $d_*(w,1,\Gamma)$ of cardinality equal to the density of the space. We relabel the family $\mathcal{F}$ as $(f_n^i)_{i\in\Gamma, n\in\mathbb{N}}$. Note that, in fact, (f_n^i) satisfies: for each $x \in d_*(w,1)\setminus\{0\}$ there is a finite set $F(x) \subset \Gamma \times \mathbb{N}$ and $\alpha(x) > 0$ such that if $(i,n) \notin F(x)$, then $|f_n^i(x)| \le \|x\| - \alpha(x)$. So the usual norm $\|\cdot\|$ of $d_*(w,1,\Gamma)$ is polyhedral and SSD. Choose x_n^i in the unit sphere $S_{\|\cdot\|}$ so that $f_n^i(x_n^i) = 1$, and relabel the biorthogonal system $\{e_i^*, e_i\}$ as $\{e_n^{*i}, e_n^i\}$. Since the weak*-$\lim_{P_F(\Gamma)} e_i^* = 0$ and weak-$\lim_{P_F(\Gamma)} e_i = 0$, we may assume, as in the proof of Proposition 3.3, that $|f_n^i(e_n^i)| < \varepsilon$ and $|e_n^{*i}(x_n^i)| < \varepsilon$. Proceeding in the same way we obtain a β-SSD norm. ∎

4. Approximation by β norms

This section is devoted to examples of some non-reflexive classical Banach spaces whose usual norm can be approximated by norms with property β. It is noteworthy here that Partington's renorming result insures the existence of *K-equivalent* norms with the mentioned property for every $K > 3$, thus it cannot be used for this purpose. On the other hand, recall that W. Schachermayer [16] proved this approximation to be true for the usual norm on ℓ_1 and for every (equivalent) norm in superreflexive spaces. Denote by δ_x the evaluation map at x.

PROPOSITION 4.1: *Let $C(K)$ be the space of continuous functions on a separable compact Hausdorff space K. The usual norm $\|\cdot\|_\infty$ can be uniformly approximated by norms satisfying property β.*

Proof: If the compact K is scattered, then we are done since $\|\cdot\|_\infty$ has property $(\beta, 0)$. Otherwise, we may choose a positive and atomless regular Borel measure $\mu \in C(K)^*$ so that $\mu(K) = 1$. Take a dense sequence $\{x_n\}_n \subset K$. Since $\mu(x_n) = 0$, there exists an open subset $U_n \subset K$ so that $x_n \in U_n$ and $\mu(U_n) \leq \varepsilon/2^{n+1}$. Then $\{x_n\}_n \subset \cup_n U_n = U$ and $\mu(U) \leq \frac{1}{2}$. In particular, $\mu(K \setminus U) \geq \frac{1}{2}$ so we may choose $y_1 \in K \setminus U$ and an open subset $V_1 \subset K$ so that $y_1 \in V_1$ and $\mu(V_1) \leq \frac{1}{4}$. We proceed by induction and obtain a sequence $\{y_n\}_n$ and open subsets $\{V_n\}_n$ in K so that $y_n \in K \setminus (U \cup V_1 \cup \cdots V_{n-1})$, $y_n \in V_n$ and $\mu(V_n) \leq 1/2^{n+1}$. Note that $\{x_n\}_n \subset U$ and $\{y_n\}_n \subset K \setminus U$ and thus every $x_n \notin \overline{\{y_n : n \in \mathbb{N}\}}$. Also, $y_n \neq y_m$ for $n \neq m$, $y_n \in V_n$ and, for $m > n$, $y_m \in K \setminus V_n$. The last implies that $y_n \notin \overline{\{y_m : m > n\}}$ and thus $y_n \notin \overline{\{y_m : m \neq n\}}$. We now proceed to construct the approximating norms. Given $\varepsilon > 0$, the family $\{\delta_{x_n} + \varepsilon\delta_{y_n}\}_n \subset C(K)^*$ will be the "dual part" of the system needed for property β. For each $n \in \mathbb{N}$, we consider the two compacts $K_n = \{x_n, y_n\}$ and $K'_n = \overline{\{y_m : m \neq n\}}$. By Urysohn's lemma, there is $f_n(x) \in S(C(K))$ so that $f_n(x_n) = f_n(y_n) = 1$ and $f_n|_{\{y_m : m \neq n\}} \equiv 0$. Clearly, $(\delta_{x_n} + \varepsilon\delta_{y_n})(f_n) = 1 + \varepsilon$ while $|(\delta_{x_n} + \varepsilon\delta_{y_n})(f_m) = |f_m(x_n)| \leq 1$ for $n \neq m$. Finally, if we define

$$|f| = \sup_n |(\delta_{x_n} + \varepsilon\delta_{y_n})(f)|$$

then $|\cdot|$ is an equivalent norm enjoying property $(\beta, \frac{1}{1+\varepsilon})$ with respect to the system $\{\delta_{x_n} + \varepsilon\delta_{y_n}, \frac{f_n}{1+\varepsilon}\}$. Moreover, $(1-\varepsilon)\|f\|_\infty \leq |f| \leq (1+\varepsilon)\|f\|_\infty$. ∎

PROPOSITION 4.2: *The usual norm $\|\cdot\|_1$ of $L_1[0,1]$ can be uniformly approximated on bounded sets by β norms.*

Proof: Let $\varepsilon \in (0,1)$ be given. Consider in $[0,1]$ two sequences $\{x_n\}, \{\varepsilon_n\} \downarrow 0$, two families of intervals $I_n = [x_n - 2\varepsilon_n, x_n + 2\varepsilon_n]$, $J_n = [x_n - \varepsilon_n, x_n + \varepsilon_n]$ such that $I_n \subset [0,1]$, $n \in \mathbb{N}$ and $I_n \cap I_m = \emptyset$, for $n \neq m$, and a dense sequence $\{f_n\} \subset S(C[0,1])$. Select $\{g_n\} \subset C[0,1]$ satisfying that $\operatorname{supp} g_n \subset I_n$, $(f_n+g_n)|_{J_n} \equiv 1+\varepsilon$ and $||f_n + g_n||_\infty = 1+\varepsilon$. Set $h_n = f_n + g_n \in C[0,1]$ and define in $L_1[0,1]$ (which embeds isometrically in $C^*[0,1]$) the equivalent norm

$$|\psi| = \sup_n |< h_n, \psi >| = \sup_n \left| \int_{[0,1]} h_n \psi \right|, \quad \psi \in L_1[0,1].$$

Clearly, $|\psi| \leq (1+\varepsilon)||\psi||$. On the other hand, $\lim_n \int_{[0,1]} g_n \psi = 0$ for every ψ. Thus, $||\psi|| \leq |\psi| \leq (1+\varepsilon)||\psi||$. Finally, note that if

$$\psi_n = \frac{1}{2\varepsilon_n} \chi_{J_n}$$

we have

$$< h_n, \psi_n >= \int_{J_n} h_n = 1+\varepsilon,$$

$$|< h_n, \psi_m >| = \left| \int_{J_m} f_n \right| \leq 1, \qquad \text{for } n \neq m$$

and then $|\cdot|$ has property $(\beta, \frac{1}{1+\varepsilon})$ with respect to the system $\{h_n, \frac{\psi_n}{1+\varepsilon}\}$. ∎

Proposition 4.2 can be generalized to every $L_1(K, \mu)$, where K is a separable Hausdorff compact space and μ is a positive regular Borel and probabilistic measure.

As we have already mentioned, apart from these examples and the cases considered by Schachermayer, there are no results on approximation with norms satisfying property β. If we fix $\varepsilon > 0$, then it is not difficult to find norms that cannot be approximated by other norms satisfying property (β, ε). For instance, it is the case for every locally uniformly rotund norm (one can be convinced by a simple drawing in the plane, playing with a circle and a polygon). Thus we can finish this note with a question that now arises in a natural way: does there exist a Banach space with a norm that cannot be arbitrarily approximated by norms with property β?

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References

[1] M. D. Acosta, F. J. Aguirre and R. Payá, *A new sufficient condition for the denseness of norm attaining operators*, The Rocky Mountain Journal of Mathematics **26** (1996), 407–418.

[2] R. Deville, V. Fonf and P. Hájek, *Analytic and polyhedral approximations of convex bodies in separable polyhedral Banach spaces*, Israel Journal of Mathematics **105** (1998), 139–154.

[3] R. Durier and P. L. Papini, *Polyhedral norms and related properties in infinite-dimensional Banach spaces: a survey*, Atti del Seminario Matematico Fisico dell' Università di Modena **40** (1992), 623–645.

[4] C. Finet and W. Schachermayer, *Equivalent norms in separable Asplund spaces*, Studia Mathematica **92** (1989), 275–283.

[5] V. Fonf, *On polyhedral Banach spaces*, Mathematical Notes of the Academy of Sciences of the USSR **45** (1989), 809–813.

[6] V. Fonf, *Three characterizations of polyhedral Banach spaces*, Ukrainian Mathematical Journal **42** No. 9 (1990), 1145–1148.

[7] C. Franchetti, *Lipschitz maps and the geometry of the unit ball in normed spaces*, Archiv der Mathematik **46** (1986), 76–84.

[8] C. Francheti and R. Payá, *Banach spaces with strongly subdifferentiable norms*, Bolletino. Unione Matematica Italiana **7-B** (1993), 45–70.

[9] G. Godefroy, *Boundaries of a convex set and interpolation sets*, Mathematische Annalen **277** (1987), 173–194.

[10] G. Godefroy, V. Montesinos and V. Zizler, *Strong subdifferentiability of norms and geometry of Banach spaces*, Commentationes Mathematicae Universitatis Carolinae **36** (1995), 493–502.

[11] M. I. Kadec and V. P. Fonf, *Two theorems on the massiveness of boundaries in reflexive Banach spaces*, Functional Analysis and its Applications **17** (1983), 227–228 (translated from Russian).

[12] V. Klee, *Polyhedral sections of convex bodies*, Acta Mathematica **103** (1960), 243–267.

[13] J. Lindenstrauss, *Norm attaining operators*, Israel Journal of Mathematics **1** (1963), 139–148.

[14] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I*, Springer-Verlag, Berlin, 1977.

[15] J. R. Partington, *Norm attaining operators*, Israel Journal of Mathematics **43** (1983), 273–276.

[16] W. Schachermayer, *Norm attaining operators and renormings of Banach spaces*, Israel Journal of Mathematics **44** (1983), 201–212.